

DIVISIBILITY GRAPH FOR SYMMETRIC AND ALTERNATING GROUPS

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ABSTRACT. Let X be a non-empty set of positive integers and $X^* = X \setminus \{1\}$. The divisibility graph $D(X)$ has X^* as the vertex set and there is an edge connecting a and b with $a, b \in X^*$ whenever a divides b or b divides a . Let $X = \text{cs}(G)$ be the set of conjugacy class sizes of a group G . In this case, we denote $D(\text{cs}(G))$ by $D(G)$. In this paper we will find the number of connected components of $D(G)$ where G is the symmetric group S_n or is the alternating group A_n .

1. INTRODUCTION

There are several graphs associated to various algebraic structures, especially finite groups, and many interesting results have been obtained recently, as for example, in [1, 5, 8, 14].

Let X be a set of positive integers and $X^* = X \setminus \{1\}$. Mark L. Lewis in [13] introduced two graphs associate with X , the common divisor graph and the prime vertex graph. The *common divisor graph* $\Gamma(X)$ is a graph with vertex set $V(\Gamma(X)) = X^*$, and edge set $E(\Gamma(X)) = \{\{x, y\} : \gcd(x, y) \neq 1\}$. Note that $\gcd(x, y)$ denotes the greatest common divisor of x, y . The *prime vertex graph* $\Delta(X)$ is a graph with vertex set $V(\Delta(X)) = \rho(X) = \bigcup_{x \in X} \pi(x)$, where $\pi(x)$ is the set of primes dividing x and edge set $E(\Delta(X)) = \{\{p, q\} : pq \text{ divides } x, x \in X\}$.

Praeger and the second author defined a bipartite graph $B(X)$ in [11] and elucidated the connection between these graphs. The *bipartite divisor graph* $B(X)$ is

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a graph with the vertex set $V(B(X)) = \rho(X) \cup X^*$, and the edge set $E(B(X)) = \{\{p, x\} : p \in \rho(X), x \in X^* \text{ and } p \text{ divides } x\}$.

Recently A. R. Camina and R. D. Camina in [6] introduced a new directed graph (or simply a digraph) using the notion of divisibility of positive numbers. The *divisibility digraph* $\vec{D}(X)$ has X^* as the vertex set and there is an arc connecting (a, b) with $a, b \in X^*$ whenever a divides b . It is clear that the digraph $\vec{D}(X)$ is not strongly connected (where by strongly connected digraph we mean a digraph such that there exists a directed path between any of two vertices). So it is important to find the number of connected components of its underlying graph. We denote the underlying graph of $\vec{D}(X)$ by $D(X)$. By the *diameter* of a graph Ω , $\text{diam}(\Omega)$, we mean the maximum diameter of its connected components.

For a finite group G and $g \in G$, let $g^G = \{x^{-1}gx : x \in G\}$ be the conjugacy class of g in G and $\text{cs}(G) = \{|g^G| : g \in G\}$ be the set of conjugacy class sizes of G . If δ is a permutation, then the cycle decomposition of δ is its expression as a product of disjoint cycles. It is interesting to investigate the properties of the prime vertex graph, the common divisor graph, the bipartite divisor graph and the divisibility graph when $X = \text{cs}(G)$. In this case we denote $\Gamma(\text{cs}(G))$, $\Delta(\text{cs}(G))$, $B(\text{cs}(G))$ and $D(\text{cs}(G))$ by $\Gamma(G)$, $\Delta(G)$, $B(G)$ and $D(G)$ respectively. For properties of $\Gamma(G)$, $\Delta(G)$ and $B(G)$ we refer to [2, 3, 8, 12].

In this paper we investigate the graph $D(G)$ where G is the symmetric group S_n or the alternating group A_n . Note that two vertices a, b of this graph are adjacent if either a divides b or b divides a . Let $\delta \in S_n$. Suppose that there are k_i cycles of length m_i ($1 \leq i \leq r$) in the cycle decomposition of δ , such that $m_i \neq 1$ and $m_i \neq m_j$, for $1 \leq i, j \leq r$, then we denote it by $\delta = [1^t, m_1^{k_1}, \dots, m_r^{k_r}]$ where $t = n - \sum_{i=1}^r k_i m_i$. Also we denote the vertex corresponding to $|g^G|$ in $D(G)$ by v_g . Throughout the paper, p is a prime number.

In Section 2 we recall some basic lemmas and theorems which we need in the next sections. In Section 3 we will find the number of connected components of $D(S_n)$. The main theorem of this section is Theorem 10. In Section 4 we will find the number of connected components of $D(A_n)$. The main theorem of this section is Theorem 16.

2. PRELIMINARIES

In this section we recall some basic technical facts that we will use later. See [4, Chapter 13], [7, Chapter 4] or [9, p.131] for proofs and details.

Lemma 1. *Suppose that $\delta = [1^t, m_1^{k_1}, \dots, m_r^{k_r}]$ where $t = n - \sum_{i=1}^r k_i m_i$, then $|C_{S_n}(\delta)| = (\prod_{i=1}^r k_i! m_i^{k_i}) t!$.*

Lemma 2. *Let $\delta \in A_n$, then there is an odd permutation in $C_{S_n}(\delta)$ if and only if $|C_{S_n}(\delta)| = 2|C_{A_n}(\delta)|$.*

Corollary 3. *Let $\delta \in A_n$. Then either $|\delta^{S_n}| = |\delta^{A_n}|$ or $|\delta^{S_n}| = 2|\delta^{A_n}|$.*

Lemma 4. *Let $\delta \in A_n$ fixes at least two points. Then $|C_{S_n}(\delta)| = 2|C_{A_n}(\delta)|$ and $|\delta^{A_n}| = |\delta^{S_n}|$.*

Lemma 5. *Suppose that $\delta = [1^t, m_1^{k_1}, \dots, m_r^{k_r}]$ and $t \leq 1$. If there exists i such that m_i is even or there exists i such that m_i is odd and $k_i \geq 2$, then $|C_{S_n}(\delta)| = 2|C_{A_n}(\delta)|$ and $|\delta^{A_n}| = |\delta^{S_n}|$.*

By Lemma 4 and Lemma 5 we have the following corollary.

Corollary 6. *If $\delta = [1^t, m_1^1, \dots, m_r^1] \in A_n$, each m_i is odd and $t \leq 1$, then we have $|C_{A_n}(\delta)| = |C_{S_n}(\delta)|$ and $|\delta^{A_n}| = \frac{1}{2}|\delta^{S_n}|$. For other cases, we have $|C_{A_n}(\delta)| = \frac{1}{2}|C_{S_n}(\delta)|$ and $|\delta^{A_n}| = |\delta^{S_n}|$.*

Lemma 7. *Let $x = \sum_{i=1}^r k_i m_i$, then $\prod_{i=1}^r k_i! m_i^{k_i}$ divides $x!$. In addition if $m_i \geq 3$, for some $1 \leq i \leq r$, then $2 \prod_{i=1}^r k_i! m_i^{k_i}$ divides $x!$.*

Proof. We prove this lemma by induction on r . Let $r = 1$, then

$$x! = \prod_{i=0}^{k_1 m_1 - 1} (k_1 m_1 - i) = k_1! m_1^{k_1} \prod_{\substack{i=1 \\ m_1 \nmid i}}^{k_1 m_1 - 1} (k_1 m_1 - i).$$

If $m_1 \geq 3$, then 2 divides $\prod_{\substack{i=1 \\ m_1 \nmid i}}^{k_1 m_1 - 1} (k_1 m_1 - i)$. Therefore $2k_1! m_1^{k_1}$ divides $x!$.

Suppose that $r = t$. So $x = \sum_{i=1}^t k_i m_i = x' + k_t m_t$ where $x' = \sum_{i=1}^{t-1} k_i m_i$. Since $\binom{x}{k_t m_t} \in \mathbb{N}$, we conclude that $x'!(k_t m_t)!$ divides $x!$. By induction hypothesis $\prod_{i=1}^{t-1} k_i! m_i^{k_i}$ divides $x'!$ and $k_t! m_t^{k_t}$ divides $(k_t m_t)!$. Let $m_i \geq 3$ for some $1 \leq i \leq r$. Then, without loss of generality, we may assume $m_t \geq 3$. So $2k_t! m_t^{k_t}$ divides $(k_t m_t)!$. Therefore $2 \prod_{i=1}^t k_i! m_i^{k_i}$ divides $x!$. \square

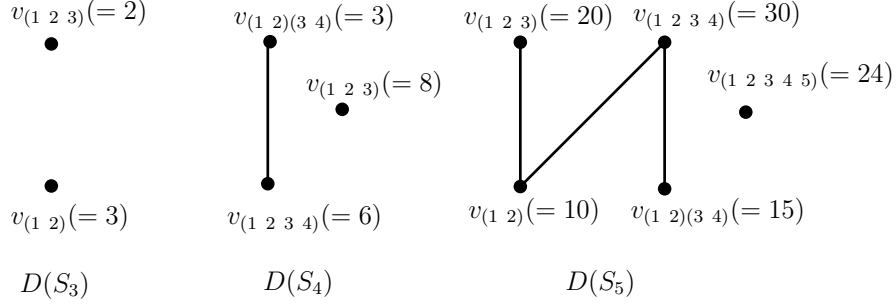
Remark 8. *Let G be a finite group and $x \in G$. It is clear that $C_G(x) \leq C_G(x^m)$ for every natural number m . So $|(x^m)^G|$ divides $|x^G|$. This means that v_x is adjacent to v_{x^m} in $D(G)$.*

3. DIVISIBILITY GRAPH FOR S_n

In this section we investigate the number of connected components of $D(S_n)$. We will prove that $D(S_n)$ has at most two connected components. If it is disconnected, then one of its connected components is an isolated vertex, that is, a copy of K_1 .

It is easy to see that both $D(S_1)$ and $D(S_2)$ are null graphs, that is, have no vertices. Also for $n = 3, 4$ and 5 , $D(S_n)$ has two connected components (see Figure 1).

Lemma 9. *Let $1 \neq \delta \in S_n$, $n > 2$ and $p \geq n - 1$. Then p divides $|C_{S_n}(\delta)|$ if and only if δ is a cycle of length p , that is, $\delta = [1, p]$ or $\delta = [p]$.*


 FIGURE 1. The graph $D(S_n)$ for $n = 3, 4$ and 5 .

Proof. First suppose p divides $|C_{S_n}(\delta)|$. Assume, to the contrary, that $\delta = [1^t, m_1^{k_1}, \dots, m_r^{k_r}]$ is not a p -cycle. By Lemma 1, $|C_{S_n}(\delta)| = (\prod_{i=1}^r k_i! m_i^{k_i}) t!$. Since p divides $|C_{S_n}(\delta)|$, then either p divides $t!$ or there exists j such that p divides m_j or $k_j!$. First assume that p divides m_j . In this case we conclude that $p = m_j$. Hence δ is a cycle of length p which is a contradiction. Now suppose that p divides $k_j!$. Since p is prime, we have $k_j \geq p \geq n - 1$. Thus $k_j m_j \geq 2n - 2 > n$, which is a contradiction too. Finally if p divides $t!$, then $t = n - \sum_{i=1}^r k_i m_i \geq p \geq n - 1$, a contradiction.

For the other direction, note that if δ is a cycle of length p then $|C_{S_n}(\delta)| = p(n - p)!$. \square

Theorem 10. *Let $1 \neq \delta \in S_n$ and $n > 6$. If δ is a p -cycle where $p \geq n - 1$, then v_δ is an isolated vertex of $D(S_n)$. The other vertices are in a single connected component.*

Proof. First suppose that δ is a cycle of length $p = n - i$ where $i \in \{0, 1\}$. Assume, to the contrary, that v_δ has a neighbor, say $v_{\delta'}$, where the cycle decomposition of δ' is not similar to δ and $|C_{S_n}(\delta')| = x$. Then $n - i$ divides x , which is a contradiction by Lemma 9. Therefore in this case v_δ is an isolated vertex of $D(S_n)$.

Let v_τ be the vertex of $D(S_n)$ corresponding to an arbitrary transposition namely τ . We prove that there exists a path between other arbitrary vertices of $D(S_n)$ and v_τ by using Lemma 1 and Lemma 7. Since for every $\delta \in S_n$ there exists a natural

number m such that $\delta^m = [1^t, p^{t'}]$, by Remark 8 it is enough to consider $\delta = [1^t, p^{t'}]$.

So we have to consider three possible cases as follows:

- (i) $\delta = [1^{n-2k}, 2^k]$ and $k \geq 2$.

Let $\delta' = [1^{n-2k}, 4^1, 2^{(k-2)}]$,

$$\frac{|C_{S_n}(\delta)|}{|C_{S_n}(\delta')|} = \frac{2^k k! (n-2k)!}{4(k-2)! 2^{k-2} (n-2k)!} = k(k-1) \in \mathbb{N}.$$

Since $|C_{S_n}(\delta')|$ divides $|C_{S_n}(\delta)|$, we conclude $|\delta^{S_n}|$ divides $|\delta'^{S_n}|$. Hence v_δ is adjacent to $v_{\delta'}$. Also by Lemma 7, there exists a positive integer s such that

$$\frac{|C_{S_n}(\tau)|}{|C_{S_n}(\delta')|} = \frac{2(n-2)!}{4(k-2)! 2^{k-2} (n-2k)!} = \frac{(n-2)!}{2^{k-1} (k-2)! (n-2k)!} = \binom{n-2}{2(k-1)} s \in \mathbb{N}.$$

So $v_{\delta'}$ is adjacent to v_τ and there is a path of length two between v_δ and v_τ .

- (ii) $\delta = [1^{n-kp}, p^k]$, $p \neq 2$ and $kp \leq n-2$.

Let $\delta' = (\alpha \beta) \delta$, where α and β are two points fixed by δ . So $\delta' = [1^{n-kp-2}, 2^1, p^k]$

and we obtain

$$\frac{|C_{S_n}(\delta)|}{|C_{S_n}(\delta')|} = \frac{k! p^k (n-kp)!}{2 \cdot k! p^k \cdot (n-kp-2)!} = \frac{(n-kp)(n-kp-1)}{2} \in \mathbb{N}.$$

Hence v_δ is adjacent to $v_{\delta'}$. Also by Lemma 7, there exists a positive integer s such that

$$\frac{|C_{S_n}(\tau)|}{|C_{S_n}(\delta')|} = \frac{2(n-2)!}{2k! p^k (n-kp-2)!} = \frac{(n-2)!}{k! p^k (n-kp-2)!} = \binom{n-2}{kp} s \in \mathbb{N}.$$

So there is a path of length 2 between v_δ and v_τ .

- (iii) $\delta = [1^{n-pk}, p^k]$, $p \neq 2$ and $kp > n-2$.

In this case we have the following three subcases:

- 1) $k \geq 3$.

Let $\delta' = [1^{n-pk}, (2p)^1, p^{(k-2)}]$,

$$\frac{|C_{S_n}(\delta)|}{|C_{S_n}(\delta')|} = \frac{k! p^k (n-kp)!}{2p(k-2)! p^{k-2} (n-kp)!} = \frac{pk(k-1)}{2} \in \mathbb{N}.$$

Again we can conclude that $|\delta^{S_n}|$ divides $|\delta'^{S_n}|$ and so v_δ is adjacent to $v_{\delta'}$. Since $p \neq 2$, we have $(k-1)p < kp-2$. Therefore $((k-1)p)!$ divides $(kp-2)!$. Hence by this fact and Lemma 7, we can find positive integers s and s' such that

$$\begin{aligned} \frac{|C_{S_n}(\tau)|}{|C_{S_n}(\delta')|} &= \frac{2(n-2)!}{2p(k-2)!p^{k-2}(n-kp)!} = \frac{(n-2)!}{(k-2)!p^{k-1}(n-kp)!} \\ &= \frac{(k-1)(n-2)!}{(k-1)!p^{k-1}(n-kp)!} = \frac{s(k-1)(n-2)!}{((k-1)p)!(n-kp)!} = \frac{ss'(k-1)(n-2)!}{(kp-2)!(n-kp)!} \\ &= ss'(k-1) \binom{n-2}{kp-2} \in \mathbb{N}. \end{aligned}$$

This means that $v_{\delta'}$ is adjacent to v_τ and there exists a path of length two between v_δ and v_τ .

2) $k = 2$.

Let $\delta' = [1^{n-p}, p^1]$. Since $p > 2$, there exists a positive integer s such that

$$\frac{|C_{S_n}(\delta')|}{|C_{S_n}(\delta)|} = \frac{p(n-p)!}{2p^2(n-2p)!} = \frac{(n-p)!}{2p(n-2p)!} = \binom{n-p}{p} s \in \mathbb{N}.$$

Thus we can conclude $|\delta'^{S_n}|$ divides $|\delta^{S_n}|$. Hence v_δ is adjacent to $v_{\delta'}$. Also $p \leq n-2$, so by case(ii) there is a path of length two between $v_{\delta'}$ and v_τ .

3) $k = 1$.

In this case δ is a p -cycle. So by Lemma 9, v_δ is an isolated vertex. \square

Corollary 11. *$D(S_n)$ has at most two connected components. If it is disconnected then one of its connected components is K_1 .*

Proof. We know that for $n \geq 6$, at most one of n or $n-1$ is a prime. By Theorem 10 and Figure 1, we obtain the result. \square

4. DIVISIBILITY GRAPH FOR A_n

In this section we consider the divisibility graph for the alternating group A_n . We will show that $D(A_n)$ has at most three connected components and if it is

disconnected then two of its connected components are K_1 . We denote $|C_{A_n}(\delta)| = (\frac{1}{2})^\sharp x$ when we do not know whether $|C_{A_n}(\delta)| = \frac{1}{2}x$ or $|C_{A_n}(\delta)| = x$ for some $x \in \mathbb{N}$.

Remark 12. *It is easy to see that $D(A_1)$, $D(A_2)$ and $D(A_3)$ are null graphs. By using GAP [10] one can see that for $n = 4, 5, 6, 7$ and 8 , $D(A_n)$ has at most three connected components (see Figure 2).*

In the rest of this section let $n \geq 9$.

Lemma 13. *Let $1 \neq \delta \in A_n$ and $p \geq n - 2$. Then p divides $|C_{A_n}(\delta)|$ if and only if δ is a cycle of length p , that is, $\delta = [1^2, p]$, $\delta = [1, p]$ or $\delta = [p]$.*

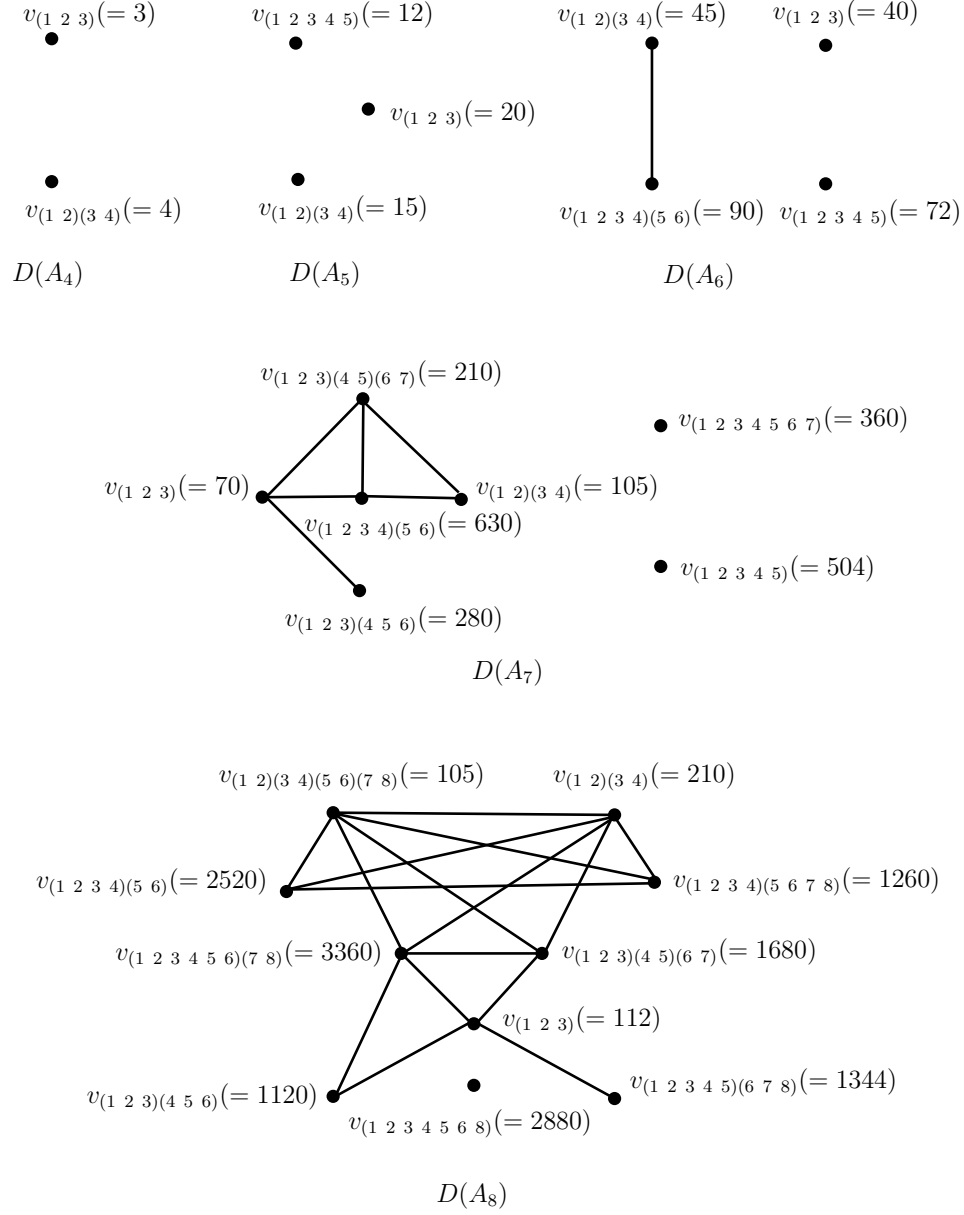
Proof. First we assume that p divides $|C_{A_n}(\delta)|$. Suppose, to the contrary, that $\delta = [1^t, m_1^{k_1}, \dots, m_r^{k_r}]$ is not a p -cycle. By Lemma 1, Lemma 2 and Corollary 6, $|C_{A_n}(\delta)| = (\frac{1}{2})^\sharp (\prod_{i=1}^r k_i! m_i^{k_i}) t!$. Since p divides $|C_{A_n}(\delta)|$ we conclude that either p divides $t!$ or there exists j such that p divides m_j or $k_j!$. If p divides either m_j or $k_j!$, then a similar argument to the proof of Lemma 9 shows that in this case either $p = m_j$ or $k_j m_j \geq 2n - 4 > n$, which is a contradiction. If p divides $t! = (n - \sum_{i=1}^r k_i m_i)!$, then $\sum_{i=1}^r k_i m_i \leq 2$. Hence δ should be a transposition, which is a contradiction too.

Now let δ be a cycle of length p . Note that p is an odd number and every cycle of length p is an even permutation. In this case $|C_{A_n}(\delta)| = (\frac{1}{2})^\sharp p(n-p)!$. Hence p divides $|C_{A_n}(\delta)|$. \square

Before proving the main theorem of this section we shall prove two lemmas.

Lemma 14. *Let $1 \neq \delta = [1^t, m_1^{k_1}, \dots, m_r^{k_r}] \in A_n$. If there exists j such that $k_j = 1$ and $m_j = 3$ then v_δ is adjacent to $v_{(1 \ 2 \ 3)}$ in $D(A_n)$.*

Proof. Without loss of generality we may assume that $k_1 = 1$ and $m_1 = 3$. Let $x = \sum_{i=1}^r k_i m_i$. By Corollary 6 and Lemma 7, there exists a positive integer s such


 FIGURE 2. The graph $D(A_n)$ for $n = 4, 5, 6, 7$ and 8 .

that

$$\begin{aligned}
 \frac{|C_{A_n}((1\ 2\ 3))|}{|C_{A_n}(\delta)|} &= \frac{\frac{1}{2} \cdot 3 \cdot (n-3)!}{(\frac{1}{2})^\sharp \cdot 3 \cdot (\prod_{i=2}^r k_i! m_i^{k_i}) (n-x)!} = \frac{(n-3)!}{(2)^\sharp (\prod_{i=2}^r k_i! m_i^{k_i}) (n-x)!} \\
 &= \binom{n-3}{x-3} s \in \mathbb{N}.
 \end{aligned}$$

Note that by Corollary 6, if 2 appears in denominator then we would have $t \leq 1$, $k_i = 1$ and m_i odd for $1 \leq i \leq r$. Since $n \geq 9$, there exists i such that $m_i \geq 5$, so by Lemma 7, we can remove “2” from the denominator.

Thus $|C_{A_n}(\delta)|$ divides $|C_{A_n}((1\ 2\ 3))|$. Therefore $|(1\ 2\ 3)^{A_n}|$ divides $|\delta^{A_n}|$. So v_δ is adjacent to $v_{(1\ 2\ 3)}$. \square

Lemma 15. *Let $1 \neq \delta = [1^t, m_1^{k_1}, \dots, m_r^{k_r}] \in A_n$. If $t \geq 3$ and for each i , $m_i \neq 3$ then there is a path of length two between v_δ and $v_{(1\ 2\ 3)}$ in $D(A_n)$.*

Proof. Let $\delta' = (\alpha\ \beta\ \gamma)\delta$, where α , β and γ are three points fixed by δ . By Corollary 6,

$$\frac{|C_{A_n}(\delta)|}{|C_{A_n}(\delta')|} = \frac{\frac{1}{2}(\prod_{i=1}^r k_i! m_i^{k_i})t!}{(\frac{1}{2})^\# \cdot 3 \cdot (\prod_{i=1}^r k_i! m_i^{k_i})(t-3)!} = \frac{t(t-1)(t-2)}{(2)^\# \cdot 3} \in \mathbb{N}.$$

This implies that $|C_{A_n}(\delta')|$ divides $|C_{A_n}(\delta)|$ and hence $|\delta^{A_n}|$ divides $|\delta'^{A_n}|$. So v_δ is adjacent to $v_{\delta'}$ and by Lemma 14, $v_{\delta'}$ is adjacent to $v_{(1\ 2\ 3)}$. \square

Theorem 16. *Let $1 \neq \delta \in A_n$. If δ is a p -cycle where $p \geq n-2$ then v_δ is an isolated vertex of $D(A_n)$. The other vertices are in a single connected component.*

Proof. First we show that if δ is a cycle of length p where $p \geq n-2$, then v_δ is an isolated vertex. Let $p = n-i$ for $i \in \{0, 1, 2\}$. Then $|C_{A_n}(\delta)| = n-i$. Suppose v_δ has a neighbor say $v_{\delta'}$, such that the cycle decomposition of δ' is not the same as δ . Let $|C_{A_n}(\delta')| = x$. In this case it is easy to see that $n-i$ divides x which is impossible by Lemma 13.

Now we are ready to show that the other vertices of $D(A_n)$ are all in the same connected component. We show that there exists a path between any other arbitrary vertex and the vertex corresponding to an arbitrary 3-cycle namely v_τ . By Lemma 1 and Corollary 6, $|C_{A_n}(\tau)| = \frac{1}{2} \cdot 3 \cdot (n-3)!$. We will use Lemma 1, Corollary 6 and Lemma 7 for our calculation. As for S_n , when $\delta \in A_n$ there is a

natural number m such that $\delta^m = [1^t, p^{t'}]$, so by Remark 8 it is enough to consider

$\delta = [1^t, p^{t'}]$. There are the following three possible cases:

(i) $\delta = [1^{n-3k}, 3^k]$ and $k \geq 2$.

If $k \geq 3$ then let $\delta' = [1^{n-3k}, 9^1, 3^{(k-3)}]$. Obviously $\delta' \in A_n$ and we obtain

$$\frac{|C_{A_n}(\delta)|}{|C_{A_n}(\delta')|} = \frac{\frac{1}{2} \cdot 3^k \cdot k! (n-3k)!}{(\frac{1}{2})^\# \cdot 9 \cdot (3^{k-3}) (k-3)! (n-3k)!} = \frac{3k(k-1)(k-2)}{(2)^\#} \in \mathbb{N}.$$

Since $|C_{A_n}(\delta')|$ divides $|C_{A_n}(\delta)|$ we conclude $|\delta^{A_n}|$ divides $|\delta'^{A_n}|$. So v_δ is adjacent

to $v_{\delta'}$. Also by Lemma 7, there exists a positive integer s such that

$$\begin{aligned} \frac{|C_{A_n}(\tau)|}{|C_{A_n}(\delta')|} &= \frac{\frac{1}{2} \cdot 3 \cdot (n-3)!}{(\frac{1}{2})^\# \cdot 9 \cdot (3^{k-3}) (k-3)! (n-3k)!} = \frac{(n-3)!}{(2)^\# \cdot 3^{k-2} (k-3)! (n-3k)!} \\ &= s \binom{n-3}{n-3k} \in \mathbb{N}. \end{aligned}$$

So $v_{\delta'}$ is adjacent to v_τ .

If $k = 2$ then,

$$\frac{|C_{A_n}(\tau)|}{|C_{A_n}(\delta)|} = \frac{3(n-3)!}{18(n-6)!} = \frac{(n-3)(n-4)(n-5)}{6} \in \mathbb{N}.$$

Again we can obtain $|\tau^{A_n}|$ divides $|\delta^{A_n}|$. So v_δ is adjacent to v_τ .

(ii) $\delta = [1^{n-kp}, p^k]$, $p \neq 3$ and $k > 1$.

Note that if $kp \leq n-3$ then δ satisfies conditions of Lemma 15. So suppose

$kp > n-3$.

We consider the following five subcases:

1) $k \geq 4$ and $p \neq 2$. In this case let $\delta' = [1^{n-kp}, (2p)^2, p^{(k-4)}] \in A_n$ and $x = kp$.

$$\frac{|C_{A_n}(\delta)|}{|C_{A_n}(\delta')|} = \frac{\frac{1}{2} k! p^k (n-kp)!}{\frac{1}{2} \cdot 2 \cdot (2p)^2 (k-4)! p^{k-4} (n-kp)!} = \frac{k! p^2}{8(k-4)!} \in \mathbb{N}.$$

This yields that $|\delta^{A_n}|$ divides $|\delta'^{A_n}|$. So v_δ is adjacent to $v_{\delta'}$.

Let $\delta'' = [1^{n-2p-3}, 3^1, p^2]$. By Lemma 7 and this fact that $p \geq 5$, we can find positive integers s and s' such that

$$\begin{aligned} \frac{|C_{A_n}(\delta'')|}{|C_{A_n}(\delta')|} &= \frac{3(n-2p-3)!}{4(k-4)!p^{k-4}(n-kp)!} = \frac{3(n-2p-3)!s}{2(kp-4p)!(n-kp)!} \\ &= \frac{3(n-2p-3)!ss'}{(kp-2p-3)!(n-kp)!} = 3ss' \binom{n-2p-3}{kp-2p-3} \in \mathbb{N}. \end{aligned}$$

So $v_{\delta'}$ is adjacent to $v_{\delta''}$. Now δ'' satisfies conditions of Lemma 14. Therefore there is a path of length three between v_{δ} and v_{τ} .

2) $k > 4$ and $p = 2$. Since $\delta \in A_n$, in this case we must have $k \geq 6$. Let $\delta' = [1^{n-2k}, (2k-2)^1, 2^1]$.

$$\frac{|C_{A_n}(\delta)|}{|C_{A_n}(\delta')|} = \frac{\frac{1}{2}.k!2^k(n-2k)!}{\frac{1}{2}.2(2k-2)(n-2k)!} \in \mathbb{N}.$$

So v_{δ} is adjacent to $v_{\delta'}$. Also let $\delta'' = [1^{n-7}, 2^2, 3^1]$.

$$\frac{|C_{A_n}(\delta'')|}{|C_{A_n}(\delta')|} = \frac{8.3.(n-7)!}{2(2k-2)(n-2k)!} \in \mathbb{N}.$$

So $v_{\delta'}$ is adjacent to $v_{\delta''}$. Now $v_{\delta''}$ satisfies conditions of Lemma 14. Therefore there is a path of length three between v_{δ} and v_{τ} .

3) $k = 4$ and $p = 2$. Let $\delta' = [1^{n-8}, 4^2]$.

$$\frac{|C_{A_n}(\delta)|}{|C_{A_n}(\delta')|} = \frac{2^4.4!(n-8)!}{4^2.2!(n-8)!} \in \mathbb{N}.$$

This means v_{δ} is adjacent to $v_{\delta'}$. Also let $\delta'' = [1^{n-4}, 2^2]$.

$$\frac{|C_{A_n}(\delta'')|}{|C_{A_n}(\delta')|} = \frac{2^2.2!(n-4)!}{4^2.2!(n-8)!} \in \mathbb{N}.$$

So $v_{\delta'}$ is adjacent to $v_{\delta''}$. By Lemma 15 there is a path of length two between $v_{\delta''}$ and v_{τ} .

4) $1 < k < 4$ and $p = 2$. In this case we have $n-2 \leq kp \leq 6$ which is a contradiction with the assumption that $n \geq 9$.

5) $1 < k < 4$ and $p \neq 2$. Since p is odd, $\delta' = [1^{n-kp+p}, p^{(k-1)}]$ is an even permutation.

In this case there exists a positive integer s such that

$$\frac{|C_{A_n}(\delta')|}{|C_{A_n}(\delta)|} = \frac{\frac{1}{2}(k-1)!p^{k-1}(n-kp+p)!}{\frac{1}{2}k!p^k(n-kp)!} = \frac{(n-kp+p)!}{kp(n-kp)!} = \binom{n-kp+p}{p}_s \in \mathbb{N}.$$

Therefore $|\delta'^{A_n}|$ divides $|\delta^{A_n}|$. So v_δ is adjacent to $v_{\delta'}$. Again according to Lemma 15 there is a path of length two between $v_{\delta'}$ and v_τ .

(iii) $\delta = [1^{n-kp}, p^k]$, $p \neq 3$ and $k = 1$.

By Lemma 13, together with our earlier assumption $kp > n - 3$, the vertex v_δ is isolated. \square

Corollary 17. *$D(A_n)$ has at most three connected components. If it is disconnected, then two of its connected components are K_1 .*

Proof. We know that for any positive integer n , at most two of the positive integers n , $n - 1$ and $n - 2$ are primes. Hence by Theorem 16 and Remark 12, we obtain the result. \square

Remark 18. *By using the fact that the distance between any vertices of $D(S_n)$ and v_τ is at most 4 (see proof of Theorem 10 and Remark 8) we can find that $\text{diam}(D(S_n)) \leq 8$. A similar argument and using the proof of Theorem 16, shows that $\text{diam}(D(A_n)) \leq 10$.*

By considering $D(S_n)$ and $D(A_n)$ for some values of n we may pose the following conjecture.

Conjecture 1. *The best upper bound for the diameter of $D(S_n)$ and $D(A_n)$ is 4.*

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